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EXCITATION OF NUCLEAR ROTATIONAL STATES IN μ-MESONIC ATOMS

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I. Introduction.

X-radiations from μ-mesonic atoms have been detected by CHANG (1949) in cosmic ray studies, and more recently by FITCH and RAINWATER (1953) working with artificially produced mesons. The mesons are captured in the outer Bohr orbits and cascade inward toward the nucleus, transferring energy by radiative and Auger transitions. The mesons arrive in the lower atomic states, for the most part, with $l = n - 1$ and proceed to decay by radiative transitions, with $\Delta l = \Delta n = -1$, to the ground state. It is the $2p$ to $1s$ transition which has been studied the most.

In the low quantum states, atomic electrons do not affect the meson and the system may be treated as a hydrogen-like atom, with due regard to the characteristic effects of the mesonic mass. The level structure of the mesonic atom has been calculated for various nuclei by WHEELER (1949, 1953), FITCH and RAINWATER, and COOPER and HENLEY (1953). Only electrical forces are considered, since the μ-meson interacts only weakly with nuclear matter. The meson is treated as a Dirac particle with spin $^1/_2$ and magnetic moment $e\hbar/2\,\mu c$. The mass μ of the μ-meson is 207* times the mass of the electron, and the Bohr orbits are proportionately smaller; in *Pb*, for example, the first mesonic Bohr orbit is 3.12×10^{-13} cm compared with a nuclear radius $R \sim 7.7 \times 10^{-13}$ cm. Nuclear structure effects, small in electronic atoms, become very pronounced in the mesonic atom. Indeed, the finite extension of the nucleus (which gives rise to isotope shifts in electronic spectra) becomes a dominant consideration in the mesonic atom; in *Pb*, for example, it accounts for the reduction of the $1s$ state energy from 21.3 MeV for a point nucleus to 10.1 MeV. This also results in a reduction of the fine structure splitting of the $2p$ doublet from 0.55 MeV to 0.2 MeV in *Pb*.

* C. f. SMITH, BIRNBAUM, and BARKAS (1953).

Nuclear moment splittings, analogous to hyperfine structure in electronic atoms, depend in general upon the expectation value of r^{-3} and are proportionately much greater in mesonic atoms. WHEELER (1953) has shown that, in the case of heavy nuclei with large distortions, the quadrupole splittings of the $2\,p_{3/2}$ level may be of the same order of magnitude as the fine structure. Magnetic h.f.s. splittings are not so greatly enhanced, since they also depend upon the mesonic magnetic moment which varies inversely as the mesonic mass; these splittings are perhaps two orders of magnitude smaller than the fine structure.

For the effects mentioned above, the meson is considered in the *static* field of the nucleus and the nucleus unaffected by the meson. COOPER and HENLEY have discussed the polarization of the nucleus by the meson—an effect treated earlier by BREIT, ARFKEN, and CLENDENIN (1950) for the electronic case—but estimated this effect to contribute not more than 3 per cent of the transition energy. Their estimate is based upon the $1s$ mesonic level where the induced monopole effect dominates; higher multipole interactions (e.g., dipole, quadrupole, etc.) for the $1s$ state involve non-diagonal matrix elements connecting mesonic states with principal quantum numbers n greater than one, and which are thus distant in energy.

In mesonic states with $n > 1$, however, the higher multipole interactions may be realized between mesonic states with the same principal quantum number, and when the nuclear excitation energy is also small, the interaction may become large. It is known from experiments on radiative lifetimes (cf. BOHR and MOTTELSON, 1953) and Coulomb excitation (cf. HUUS and ZUPANČIČ, 1953) that in nuclei with large deformations there exist low-lying excited states which have very large quadrupole transition probabilities to the ground state. For these nuclei, the excitation matrix elements are comparable in magnitude with the static quadrupole interactions discussed above, and in many cases are larger than the nuclear excitation energies. The non-static meson-nuclear quadrupole interaction must thus be expected to have a major influence on the "fine structure" of the mesonic X-rays. It also provides a large probability that, after the meson reaches the atomic ground state, the nucleus be left in an excited state and subsequently emit a nuclear γ-ray.

These effects occur for even-even as well as for odd nuclei, since, although the quadrupole interaction vanishes in the ground state of even-even nuclei ($I = 0$), it does not vanish generally in the excited states, and non-zero matrix elements connect the ground state with excited states.

The effects of the interaction are of interest not only in understanding the spectra, but especially in providing another method of obtaining information about the magnitude and sign of nuclear deformations and nuclear charge distributions. The mesonic atom may, in fact, provide the first method of determining the sign of intrinsic quadrupole moments in even-even nuclei.

II. Description of the Model.

The low-lying states which interact especially strongly with the meson follow a very regular pattern and have been rather accurately described in terms of rotational states of intrinsically deformed nuclei (Bohr, 1952, 1954; Bohr and Mottelson, 1953). The theory of these states accounts for their nuclear energy spectra and also predicts many simple relations between matrix elements connecting the rotational states. These relations make it possible to greatly reduce the parameters entering into the calculations of the effects of the meson-nuclear interaction, and to interpret the empirical data of these effects in terms of simple nuclear properties. In the following, we shall describe the effects of the interaction of the meson with the excited nuclear states in terms of the model of the rotational states, even though the considerations involved in the calculation of the interaction are more generally valid.

Rotational spectra are expected in nuclei with large deformations, and have been observed to occur with considerable regularity in nuclei with $155 < A < 185$ and $A > 255$. Such nuclei may be described as possessing an intrinsic deformation which is usually symmetric about some nuclear axis. The rotation of the deformed nucleus is generated by a collective motion of the nucleons, which is similar to the classical motion of an irrotational fluid. The rotation leaves the nuclear shape unaltered and affects only the orientation of the nuclear axis. The state of the nucleus may be specified by quantum numbers I, M_I and K—the total nuclear angular momentum, the projection along the z-axis, and the projection along the symmetry axis. In the low-lying rotational states, K is a constant and is just equal to the projection of the angular momenta of the individual nucleons along the symmetry axis; for the ground state, $I_0 = K$ except when $K = 1/2$.*

The rotational spectrum is given by

$$H_{ROT} = \frac{\hbar^2}{2\mathfrak{J}}\left[I(I+1) - I_0(I_0+1)\right]. \tag{1}$$

For odd-A or odd-odd nuclei, the sequence of states is $I = I_0$, $I_0 + 1$, $I_0 + 2$, ... while for even-even nuclei, where $I_0 = K = 0$, the reflection symmetry of the nuclear shape implies that only even integral values of I occur.

The moment of inertia, $\mathfrak{J}$, of such a system may be shown to be proportional to the square of the deformation, and the wave functions to be the properly symmetrized (with respect to the sign of K) symmetric top functions. The symmetrization depends upon the nucleonic configurations, and plays no essential role in the present discussion. For our present purposes**, it is thus sufficient to consider the unsymmetrized nuclear wave functions

$$\psi^I_{M_I K} = \sqrt{\frac{2I+1}{8\pi^2}}\,\mathfrak{D}^I_{M_I K}(\theta i), \tag{2}$$

where the θi are the three Eulerian angles (θ, φ, ψ) describing the orientation of the nuclear axes. For even-even nuclei, we have $K = 0$, and the wave functions are then given more simply by

$$\psi^I_{M_I O} = \sqrt{\frac{2I+1}{8\pi^2}}\,\mathfrak{D}^I_{M_I O}(\theta i) = \frac{1}{\sqrt{2\pi}}\,Y^I_{M_I}(\theta,\varphi). \tag{2 a}$$

The nuclear distortion is characterized by an intrinsic quadrupole moment, Q_0, oriented along the symmetry axis. It is related

* In the case $K = 1/2$, the spectrum is modified by the inclusion of terms which depend upon the nucleonic structure (cf. Bohr and Mottelson, 1953, eq. II. 24). The wave functions are, however, still of the form (2) and the discussion here does not depend upon the form of eq. (1).

** In section IV, where strong coupling of the meson to the nucleus is discussed, we will require an expression for the symmetrized wave functions. Equation (16) is thus analogous to (2) where the mesonic wave functions replace the nucleonic wave functions implied in (2). Cf. Bohr and Mottelson for a complete discussion.

to the spectroscopically observed quadrupole moment Q by the relation

$$Q = \frac{3\,K^2 - I(I+1)}{(I+1)(2\,I+3)}\,Q_0, \tag{3}$$

which vanishes for $K = I = 0$ or $1/2$.

The interaction of the meson with the rotational states arises from the deviation of the nuclear field from spherical symmetry due to the deformation. The most important interaction is due to the quadrupole coupling and may be written in the form

$$H' = -\tfrac{1}{2}\,Q_0 e^2 f(r)\,P_2(\cos\widehat{\mu N}), \tag{4}$$

where $\widehat{\mu N}$ is the angle between the meson radius vector and the symmetry axis of the nucleus. The function $f(r)$, where r is the radial coordinate of the meson, contains the radial dependence of the interaction, and is given in general form by

$$Q_0 f(r) = r^{-3}\int_0^r \varrho^{(2)}(\vec{r}\,')\,dv' + r^2\int_r^\infty r^{-5}\varrho^{(2)}(\vec{r}\,')\,dv', \tag{5}$$

where the primed coordinates refer to the nucleus and $\varrho^{(2)}(\vec{r}\,') = (3\,z'^2 - r'^2)$ is the quadrupole part of the nuclear charge distribution and is oriented along the nuclear symmetry axis. The first integral in (5) gives just Q_0 when extended over the entire charge of the nucleus. If the quadrupole distribution were concentrated at the surface, the radial function could be written as

$$f(r) = \begin{cases} r^{-3} & \text{if } r > R \\ r^2 R^{-5} & \text{if } r < R, \end{cases} \tag{6}$$

where R is the nuclear radius. This may also be used as an approximation for a uniformly charged ellipsoidal nucleus (cf. WHEELER, 1953).

III. Treatment of Coupled System of Meson and Rotating Nucleus.

The energy of the coupled system is given by

$$H = H_0(\vec{x}, \vec{\sigma}) + H_{ROT}(\theta i) + H'(\vec{x}, \theta_i), \tag{7}$$

where $\vec{x}$ and $\vec{\sigma}$ are the mesonic space and spin coordinates, and $H_0(\vec{x}, \vec{\sigma})$ is the meson energy in the nuclear monopole field (i.e. the electrostatic potential averaged over all angles). This energy includes the fine structure splittings, and is characterized by the quantum numbers n, l (approximately), and j.

We shall consider the equations of motion in the uncoupled representation. We thus choose as basis vectors products of the nuclear wave functions $\Psi^I_{M_I K}$ and the meson eigenfunctions of the monopole potential. The total wave functions of the system characterized by F, the total angular momentum and M, the projection along the z-axis, may then be denoted by

$$|IK, j; FM\rangle = \sum_{M_I m} (IjM_I m | IjFM)\, \psi^I_{M_I K}(\theta_i)\, \varphi^j_m(\vec{x}, \vec{\sigma}), \tag{8}$$

where $(IjM_I\, m | IjFM)$ is the Clebsch-Gordon coefficient for adding angular momenta. Both H_0 and H_{ROT} are diagonal in this representation.

The Legendre polynomial $P_2(\cos \widehat{\mu N})$ which appears in H' may be written as a scalar product of spherical harmonics of order two in the meson and nuclear orientation angles:

$$P_2(\cos \widehat{\mu N}) = \frac{4\pi}{5} \sum_{q=-2}^{2} (-1)^q\, Y_{2q}(\theta_N \varphi_N)\, Y_{2-q}(\theta_\mu \varphi_\mu). \tag{9}$$

The matrix elements for the interaction may then be written in the form (cf. Racah (1942) whose notation we follow)

$$\left. \begin{aligned} \langle IK, j; F | H' | I'K', j'; F'\rangle &= -\tfrac{1}{2} Q_0 e^2 \frac{4\pi}{5} (-1)^{I-j'-F} W(IjI'j'; F2) \\ &\times \langle IK \| Y_N^{(2)} \| I'K\rangle \langle j \| Y_\mu^{(2)} \| j'\rangle \langle j \| f(r) \| j'\rangle\, \delta_{FF'} \delta_{KK'}. \end{aligned} \right\} \tag{10}$$

The dependence of the matrix elements on F is contained only in the Racah coefficients, which may be evaluated from tables*. The double-bar matrix elements depend only on the nuclear or mesonic configurations. The nuclear matrix elements are given by

$$\langle IK \| Y_N^{(2)} \| I'K \rangle = \sqrt{\frac{5}{4\pi}}(2I'+1)(2I'OK|2I'IK) \qquad (11)$$

from which it is clear that $|I - I'| \leqslant 2$.

The meson matrix elements depend implicitly upon other quantum numbers, in particular n and l. The element $\langle j \| Y_\mu^{(2)} \| j' \rangle$ vanishes for $j = j' = 1/2$, and since $Y_\mu^{(2)}$ is of even parity, it can only connect mesonic states of the same parity. States with different values of n are too far away in energy to mix. Thus the $1s_{\frac{1}{2}}$ and $2s_{\frac{1}{2}}$ states are unaffected by the interaction, and $2p$ states can mix only among themselves. With increasing n, the effect of the interaction rapidly decreases and is already rather small for $n = 3$ (cf. footnote on p. 14). The non-vanishing angular matrix elements for the p-states are given by

$$\left.\begin{aligned} \langle 3/2 \| Y_\mu^{(2)} \| 1/2 \rangle &= -\langle 1/2 \| Y_\mu^{(2)} \| 3/2 \rangle \\ &= -\langle 3/2 \| Y_\mu^{(2)} \| 3/2 \rangle = (\pi)^{-1/2}. \end{aligned}\right\} \qquad (12)$$

The radial double-bar matrix elements

$$\langle j \| f(r) \| j' \rangle = \int_0^\infty \Re_{2pj}(r) f(r) \Re_{2pj'}(r)\, dr \qquad (13)$$

must be evaluated from a knowledge of radial wave functions. Wheeler (1953) approximates the integral from hydrogenic Schroedinger wave functions for the $2p$ states, $\Re_{2p} = c^2 r^2 \exp(-Z\mu e^2 r/2\hbar^2)$, normalized so that $\int \Re_{2p}^2 dr = 1$, and using (5) for $f(r)$. He then finds

$$e^2 \int \Re_2^{2p} f(r)\, dr = 5\,(Z/237)^3 f_q \text{ Mev/barn}, \qquad (14)$$

where the form factor $f_q \approx (1 + 0.1x^2)^{-2}$ and the dimensionless parameter $x = RZ\mu e^2/\hbar^2$. Although the hydrogenic Schroedinger

* Cf., for example, Biedenharn (1952); Biedenharn, Blatt, and Rose (1952), or Simon, van der Sluis, and Biedenharn (1954).

wave functions may be considered poor approximations for a $2p$ meson and a Pb nucleus (the second Bohr orbit is 12.3×10^{-13} cm compared with $R \sim 7.7 \times 10^{-13}$ cm for Pb), the wave functions published by FITCH and RAINWATER for Pb do yield a form factor only 6 per cent smaller than that predicted by WHEELER.

Because F is a good quantum number, the energy matrix is reducible to submatrices in which F is a constant. When these submatrices are diagonalized, the new eigenfunctions are linear combinations of the functions given in (8); we may write them in the form

$$\left|\alpha, K; FM\right> = \sum_{Ij} \left|IK, j; FM\right> \left<IK, j; F\right|\alpha, K; F\right>, \qquad (15)$$

where α designates the other quantum numbers necessary to specify the particular state.

As will be shown in Section V, the only states which are populated with appreciable intensity are those which contain a component having the nucleus in its ground states. The number of these states is given in Table 1.

TABLE 1.

Even-Even Nuclei		Odd Nuclei		
F	No. of levels	F	No. of levels	
1/2	2	$I_0 - 3/2$	1	$(I_0 \geqslant 3/2)$
3/2	3	$I_0 - 1/2$	3	
		$I_0 + 1/2$	5	
		$I_0 + 3/2$	6	

This is to be constrasted with the case of no non-diagonal interactions, where only a single state for each of the listed F-values is populated.

The number of states for each value of F indicates the order of the matrix which must be diagonalized. The procedure for diagonalization is straightforward, but in the case of matrices of order 5 and 6 numerical approximation methods must be used.

The rotational spectra of the simple type (1) represent a limiting case realized for very deformed nuclei. For less deformed nuclei, the excitation spectrum is less regular but, provided the essential non-diagonal coupling results from the interaction with one or a few low-lying states, its effect can be analyzed in a somewhat similar manner as above. The result will depend on certain matrix elements which represent partly the average quadrupole coupling of the meson with the nucleus in its ground state and excited states, and partly the quadrupole transition elements which are similar to those which determine the electric quadrupole radiative transitions between the states in question. In general, however, one can expect no simple relationship between the various matrix elements such as characterizes the rotational spectrum.

IV. Strong Coupling Approximation.

For sufficiently strong meson-nuclear quadrupole interaction, it is possible to obtain a simple solution to the coupled equations. Such a strong coupling treatment has been developed by BOHR and MOTTELSON (1953) for coupling nucleons to a deformed nucleus, and the methods are applicable also to a meson in the nuclear quadrupole field.

In the strong coupling treatment, one considers the meson as moving relatively to the deformed nucleus, and the appropriate basis vectors are thus given by

$$\left.\begin{aligned} |K, j\Omega_\mu; FM\rangle_s = \sqrt{\frac{2F+1}{16\pi^2}}\,&[\mathfrak{D}^F_{MK+\Omega_\mu}(\theta_i)\,\varphi^j_{\Omega_\mu}(\vec{x}', \vec{\sigma}') \\ &+(-1)^{F-j}\mathfrak{D}^F_{M-(K+\Omega_\mu)}(\theta_i)\,\varphi^j_{-\Omega_\mu}(\vec{x}', \vec{\sigma}')], \end{aligned}\right\} \tag{16}$$

in which the mesonic wave functions are described in terms of coordinates relative to the nuclear symmetry axis. The quantum number Ω_μ represents the component of j along this symmetry axis. The sign of the symmetrization is that appropriate to a meson coupled to an even-even nucleus. For odd-A or odd-odd nuclei, the symmetrization is of no significance in the present context.

The matrix elements of the quadrupole interaction H' (4) become very simple in this representation since H' is equivalent to the interaction of a meson in a fixed quadrupole field (or to a meson in the field of a nucleus with $I = \infty$). The elements are given by

$$\left.\begin{array}{l} \langle K, j\Omega_\mu; F \,|\, H' \,|\, K', j'\Omega'_\mu; F'\rangle = -\tfrac{1}{2} Q_0 e^2 \langle j \,\|\, f(r) \,\|\, j'\rangle \\ \times \frac{\sqrt{4\pi}}{5} (-1)^{j'-\Omega_\mu} (jj'\Omega_\mu - \Omega_\mu \,|\, jj'\, 20) \langle j \,\|\, Y_\mu^{(2)} \,\|\, j'\rangle\, \delta_{FF'} \delta_{KK'} \delta_{\Omega_\mu \Omega'_\mu} \end{array}\right\} \quad (17)$$

and are seen to be diagonal in Ω_μ.

The rotational energy of the system possesses non-diagonal as well as diagonal matrix elements in Ω_μ. The diagonal terms are given by (cf. Bohr and Mottelson, 1953, Eq. II. 24)

$$\left.\begin{array}{l} (H_{ROT})_0 = \frac{\hbar^2}{2\mathfrak{J}} \{F(F+1) + j(j+1) - I_0(I_0+1) - 2\,\Omega_\mu (K + \Omega_\mu) \\ \qquad - (-1)^{F-j} (j+1/2)(F+1/2)\, \delta_{\Omega_\mu, \frac{1}{2}} \delta_{K,O}\}. \end{array}\right\} \quad (18)$$

The last term in (18), which is to be included only for a meson coupled to an even-even nucleus, arises from symmetrization of the wave function (16), which introduces additional diagonal terms.

The matrix elements of the rotational energy which are non-diagonal in Ω_μ tend to decouple the meson from the nuclear axis, and the strong coupling approximation is the neglect of these terms. This approximation is valid when the rotational energies are small compared with the quadrupole energies (17).

The strong coupling Hamiltonian, $H_0 + (H_{ROT})_0 + H'$, is thus diagonal in Ω_μ as well as in F, K and M, and the eigenfunctions are linear combinations (with respect to j) of functions of the type (16),

$$|\,\alpha, K, \Omega_\mu; FM\rangle_s = \sum_j |\,K, j\Omega_\mu; FM\rangle_s \langle K, j\Omega_\mu; F \,|\, \alpha, K, \Omega_\mu; F\rangle_s. \quad (19)$$

If we consider the $2\,p$ states, the sum in (19) for $\Omega_\mu = \pm 3/2$ contains just one state with $j = 3/2$, while for $\Omega_\mu = \pm 1/2$ it contains two states with $j = 1/2$ and $3/2$. Thus, the diagonalization procedure involves at most matrices of order two. It is convenient (e.g. when discussing line intensities) also to express the strong coupling wave functions (19) in the form of the uncoupled representation (15). This transformation is given by

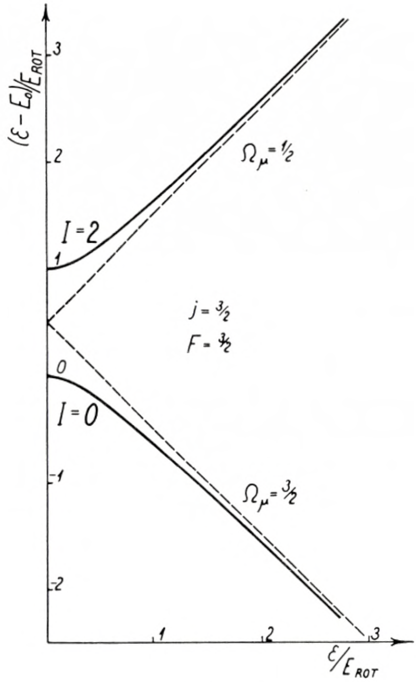

FIGURE 1.

A comparison is made between the energies given by the strong coupling approximation and the "exact" treatment for an even-even nucleus. It is assumed that the meson is in a $2p$ state, and that the fine structure splitting is large (j is a good quantum number), in which case, for the $F = 1/2$ levels and the $F = 3/2$, $j = 1/2$ level, the strong coupling approximation reduces to the exact treatment. The solid curves give the "exact" energies and the dashed curve the strong coupling limit for the levels $F = 3/2$, $j = 3/2$. E_0 is the energy in the absence of quadrupole coupling and E_{ROT} is the excitation energy of the first rotational state ($I = 2$). The quadrupole coupling is expressed in terms of

$$\varepsilon = -\tfrac{1}{10} Q_0 e^2 \langle j \,||\, f(r) \,||\, j' \rangle.$$

In the region of weak coupling, ($\varepsilon/E_{ROT} \ll 1$), the nuclear spin I is approximately a good quantum number, while in the region of strong coupling ($\varepsilon/E_{ROT} \gg 1$), the component of the meson angular momentum along the nuclear symmetry axis, Ω_μ, is approximately a good quantum number.

$$\left.\begin{array}{l} \left|\alpha, K, \Omega_\mu; FM\right\rangle_s = \sum\limits_{Ij} \left| IK, j; FM\right\rangle \left\langle K, j\Omega_\mu; F \right| \alpha, K, \Omega_\mu; F\rangle_s \\ \times (IjK\Omega_\mu \,|\, Ij\,FK+\Omega_\mu) \sqrt{\dfrac{2I+1}{2F+1}} \times \left\{ \begin{array}{ll} 1 & \text{for odd-}A \text{ or odd-odd nuclei} \\ 0 \quad \text{for } I \text{ odd} & \multirow{2}{*}{$\left.\right\}$ even-even nuclei.} \\ \sqrt{2} \text{ for } I \text{ even} & \end{array} \right. \end{array} \right\} \quad (20)$$

A measure of the error in the energy eigenvalues of the $2\,p$ levels is given by

$$\delta E \approx \frac{1}{8} \frac{(E_{ROT})^2}{\frac{1}{2} Q_0 e^2 \langle 1/2 \,\|\, f(r) \,\|\, 3/2 \rangle}, \tag{21}$$

where E_{ROT} is the energy of the first rotational state; the numerical coefficient refers to even-even nuclei. Fig. 1 compares the strong coupling and exact energies for $F = j = 3/2$ levels in an even-even nucleus; the fine structure splittings are assumed to be very large so that j is a good quantum number and thus the three other levels are given exactly in strong coupling.

Although the error in the energy is quadratic in E_{ROT} (cf. Eq. (21)), the error in the wave function is linear in E_{ROT}. Thus, the strong coupling approximation should be used with reservation when wave functions are required (e.g. for line intensities). The exact treatment is always available and offers no fundamental difficulties.

V. Line Intensities.

The question of the X-ray line intensities for the $2\,p$—$1\,s$ mesonic transitions involves an investigation of two points: (1) the relative (rate of) population of the $2\,p$ levels, and (2) the relative transition probabilities from the $2\,p$ states to the $1\,s$ ground states.

The radiative transitions of interest are of the *atomic* electric dipole type. The nuclear transitions, which are $M1$ or $E2$, are several orders of magnitude slower. If I were a good quantum number, we would therefore have the restriction $\Delta I = 0$. When I is not a good quantum number, the electric dipole matrix element vanishes between those components of the wave function for which $I_i \neq I_f$.

The $2\,p$ states are populated from higher states which interact only weakly with the nuclear quadrupole field and are, therefore, very nearly pure $I = I_0$*. The populations of the $2\,p$ states (i. e. summed over M) can be shown to be proportional to

$$(2\,F + 1) \sum_j \langle I_0 K, j;\, FM \,|\, \alpha, K;\, FM \rangle^2, \tag{22}$$

where the $\langle\, IK, j;\; FM \,|\, \alpha, K;\; FM \,\rangle$ are defined by (15).

* Of these, the $3d$ states are most responsible for feeding the $2p$ states, and also interact the most with the nucleus. Mixing of excited nuclear states becomes appreciable for $n = 3$ only for the nuclei with large deformations beyond Pb, and then may affect the intensities by around 20 per cent. The effect of mixing in the $3d$ states may be included in a straightforward manner.

In the mesonic ground state, $1s_{1/2}$, there is no interaction with the nuclear quadrupole field, and F, I and $j = 1/2$ are all good quantum numbers; the energy depends only upon I. The relative transition probabilities from the $2p$ level $|\alpha, K; F\rangle$ to the ground level with spin I are proportional to

$$\sum_j \langle IK, j; FM \,|\, \alpha, K; FM\rangle^2. \tag{23}$$

From (22) and (23), the line intensities can be computed. An atom which finds itself in the ground state, but with $I > I_0$, will emit a nuclear γ-ray. For nuclei with large deformations, the probability for this may be of the order of 1/2.

What might be called the "center of population" of the $2p$ states is left unchanged by the inclusion of the quadrupole interaction. However, the possibility of making transitions to the atomic ground states with an excited nucleus tends to shift the center of gravity of the spectral lines to smaller energies, and this shift is just given by weighting the nuclear rotational energies with number of transitions to these final states. Such shifts will be less than about 1 per cent of the transition energies.

VI. Numerical Examples.

It is to be noted that the quadrupole moment enters only in the combination $Q_0 \langle j \| f(r) \| j'\rangle$. In working out numerical examples, we shall select values for this combination which are consistent with other estimates of nuclear quadrupole moments and with the assumptions that the double-far matrix element may be approximated by (14). The fine structure splittings are roughly interpolated from the numerical values of Fitch and Rainwater. The other parameters which enter are indicated in the particular cases.

A. Even-Even Nuclei. The ground state of an even-even nucleus has $I = 0$ and the first rotational level is $I = 2$, $K = 0$. No higher levels enter for the $2p$ states, and the Hamiltonian is reducible to the submatrices given below.

FIGURES 2.

Energy levels and spectra are given for even-even isotopes. All energies are given in MeV. In each of the three examples are three diagrams which give, beginning at the top,

(I) 2 p energy level scheme in the absence of non-diagonal interactions. The triplet of numbers above each line designates (IjF), all of which are good quantum numbers when the non-diagonal interactions are neglected. The solid lines represent the levels where the nucleus is in the ground state and give the usual fine structure doublet; these are the only levels which are populated when no "mixing" is present. The dashed lines represent atomic levels in which the first nuclear rotational level ($I = 2$) is excited, and includes diagonal (static) quadrupole interactions. The height of the lines is proportional to the statistical weight ($2F+1$). The zero of energy is taken at the "center of gravity" of the fine structure doublet ($I = 0$).

(II) 2 p energy level scheme including non-diagonal as well as diagonal quadrupole interactions. The height of each line is proportional to its population. The only good quantum number which remains after "mixing" is F, which is denoted under each line.

(III) The line spectrum. Each of the $2p$ states may make transitions to the atomic ground states, $1s_{1/2}$ with $I = 0$ or 2. This leads to the ten spectral lines represented by the solid lines. The height of the lines is proportional to the intensity. The dashed lines represent the spectrum which would be observed in the absence of non-diagonal interactions; the zero point of energy is taken at the center of gravity of this doublet. The arrow points to the center of gravity of the actual spectrum.

The values of the parameters represent estimates which contain considerable uncertainty, but are expected to exhibit the salient features of the spectra. The fine structure energies (E_{fs}) are rough extrapolations of the values given by FITCH and RAINWATER. The rotational energies (E_{ROT}) are either experimental values or consistent with the energies of neighbouring isotopes. The quadrupole interaction energy

$$\varepsilon = -\frac{1}{10} Q_0 e^2 \langle j \,||\, f(r) \,||\, j' \rangle \approx -\frac{1}{2} Q_0 (Z/237)^3 f_q \text{ MeV}$$

is based upon Q_0 consistent with spectroscopic data (from neighbouring odd isotopes), nuclear rotational energies, and Coulomb excitation. The form factor f_q is given by Wheeler's approximation. The actual parameters are given below each example.

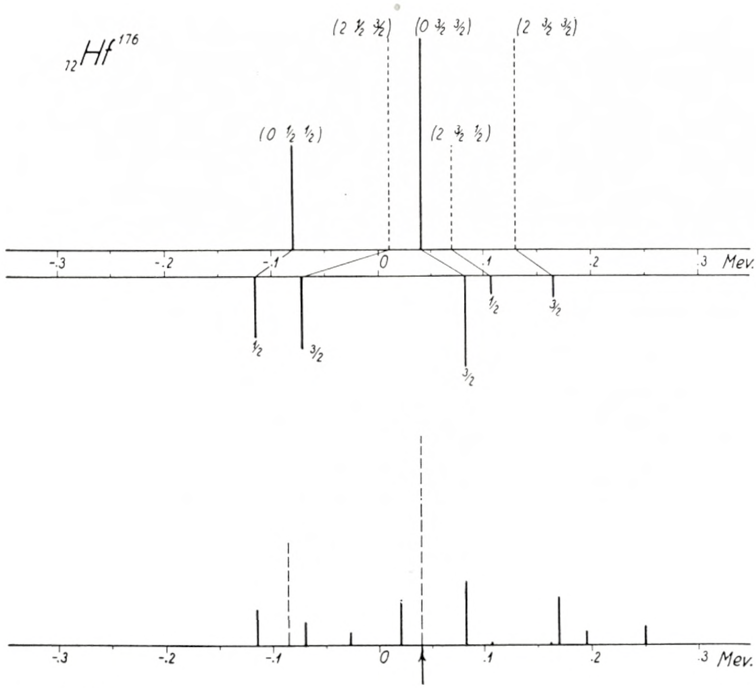

FIG. 2a, $_{72}Hf^{176}$:

$$E_{fs} = 0.119 \text{ MeV},$$
$$E_{ROT} = 0.089 \text{ MeV},$$
$$\varepsilon = 0.0579 \text{ MeV } (Q_0 \approx 9 \text{ barns}, f_q \approx 0.457).$$

Probability that the nucleus be left in the state $I = 2$ is 0.47.

$$\left.\begin{array}{l} \begin{pmatrix} H_{11} & \sqrt{2}\,\varepsilon \\ \sqrt{2}\,\varepsilon & H_{22} \end{pmatrix} \begin{array}{l} \cdots \left| 00, 1/2, 1/2\, M \right\rangle \\ \cdots \left| 20, 3/2, 1/2\, M \right\rangle \end{array} \end{array}\right\} \tag{24}$$

and

$$\left.\begin{array}{l} \begin{pmatrix} \bar{H}_{11} & -\varepsilon & \varepsilon \\ -\varepsilon & \bar{H}_{22} & \varepsilon \\ \varepsilon & \varepsilon & \bar{H}_{33} \end{pmatrix} \begin{array}{l} \cdots \left| 00, 3/2;\, 3/2\, M \right\rangle \\ \cdots \left| 20, 1/2;\, 3/2\, M \right\rangle \\ \cdots \left| 20, 3/2;\, 3/2\, M \right\rangle \end{array} \end{array}\right\} \tag{25}$$

where

$$H_{11} = H_{00} - \tfrac{2}{3} E_{fs}$$
$$H_{22} = H_{00} + \tfrac{1}{3} E_{fs} + E_{ROT} + \varepsilon$$

$$\bar{H}_{11} = H_{00} + \tfrac{1}{3} E_{fs}$$
$$\bar{H}_{22} = H_{00} - \tfrac{2}{3} E_{fs} + E_{ROT}$$
$$\bar{H}_{33} = H_{00} + \tfrac{1}{3} E_{fs} + E_{ROT}$$

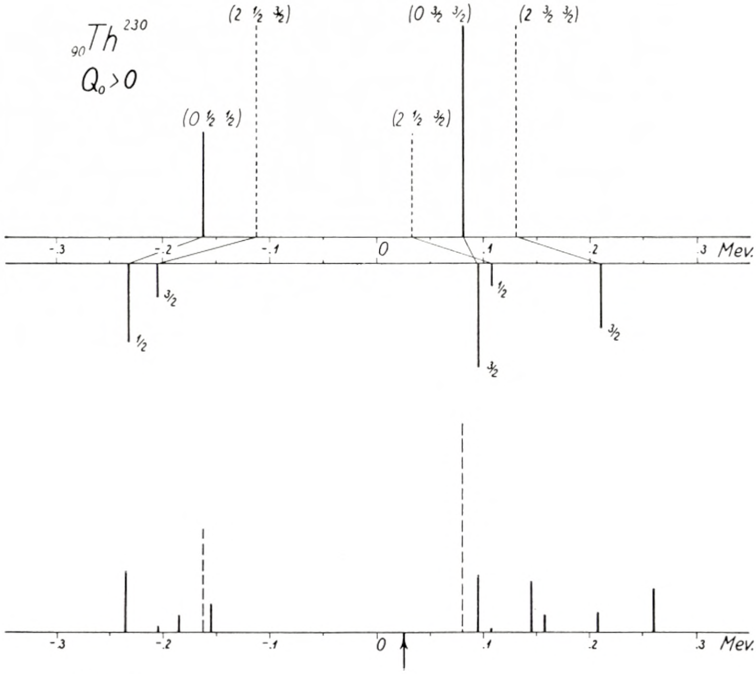

FIG. 2*b*, $_{90}Th^{230}$, $Q_0 > 0$:

$$E_{fs} = 0.242 \text{ MeV},$$
$$E_{ROT} = 0.050 \text{ MeV},$$
$$\varepsilon = 0.0986 \text{ MeV } (Q_0 \approx 12.6 \text{ barns}, f_q \approx 0.286).$$

Probability that the nucleus be left in the state $I = 2$ is 0.52.

and H_{00} = "center of gravity" of the unperturbed $2\,p$ doublet

$$E_{fs} = \text{fine structure splitting } 2\,p_{3/2} - 2\,p_{1/2}$$
$$E_{ROT} = \text{energy of first rotational nuclear level}$$
$$\varepsilon = -\tfrac{1}{10}\,Q_0 e^2 \langle j \,\|\, f(r) \,\|\, j' \rangle \approx -\tfrac{1}{2}\,Q_0 (Z/237)^3 f_q \text{ MeV}.$$

In Figs. 2, the level structure and line intensities are given for even-even isotopes *Hf* and *Th*. The parameters assumed are described in the captions. The spectra are quite different from those which are anticipated without inclusion of the non-diagonal interaction, even if the individual lines are not resolvable. Of particular interest is the way in which the spectra clearly distinguish the sign of the intrinsic quadrupole moment. In the *Th* example, the negative sign leads to three well-separated groups of lines, the positive sign to two.

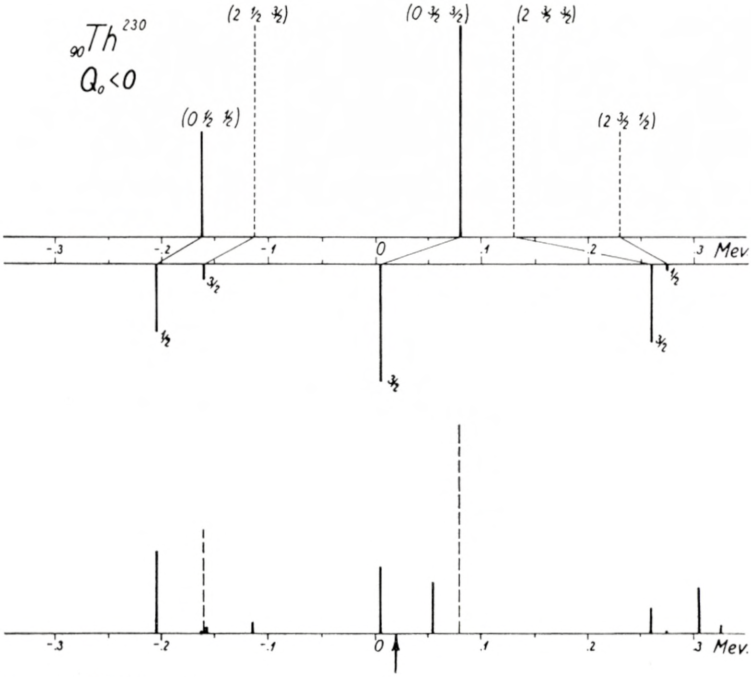

FIG. 2c, ${}_{90}Th^{230}$, $Q_0 < 0$:
All parameters are the same as for (2b), except that Q_0 and hence ε are negative. Probability that the nucleus be left in the state $I = 2$ is 0.42.

B. Odd Nuclei. As is indicated in Table 1, the odd nuclei will, in general, lead to 15 levels and require, for calculation, the diagonalization of matrices of order up to six. Although this is straightforward, we choose, for the numerical example, to use the strong coupling approximation and select a nucleus for which it is likely to be valid, U^{235} ($I = 5/2$).

Since $H_0 + (H_{ROT})_0 + H'$ is reducible in Ω_μ, the states with $\Omega_\mu = \pm 1/2$ are obtained by diagonalizing the 2×2 matrices

$$\left.\begin{matrix} \begin{pmatrix} H_{11} & \sqrt{2}\,\varepsilon \\ \sqrt{2}\,\varepsilon & H_{22} \end{pmatrix} & \begin{matrix} \ldots \left| 5/2,\, 1/2 \pm 1/2;\; FM \right\rangle_s \\ \ldots \left| 5/2,\, 3/2 \pm 1/2;\; FM \right\rangle_s, \end{matrix} \end{matrix}\right\} \quad (26)$$

where

$$H_{11} = H_{00} - \tfrac{2}{3} E_{fs} + \frac{\hbar^2}{2\,\Im}\{F(F+1) - 17/2 \mp 5/2\}$$

$$H_{22} = H_{00} + \tfrac{1}{3} E_{fs} + \varepsilon + \frac{\hbar^2}{2\,\Im}\{F(F+1) - 11/2 \mp 5/2\}$$

and ε is defined as in expressions (24) and (25).

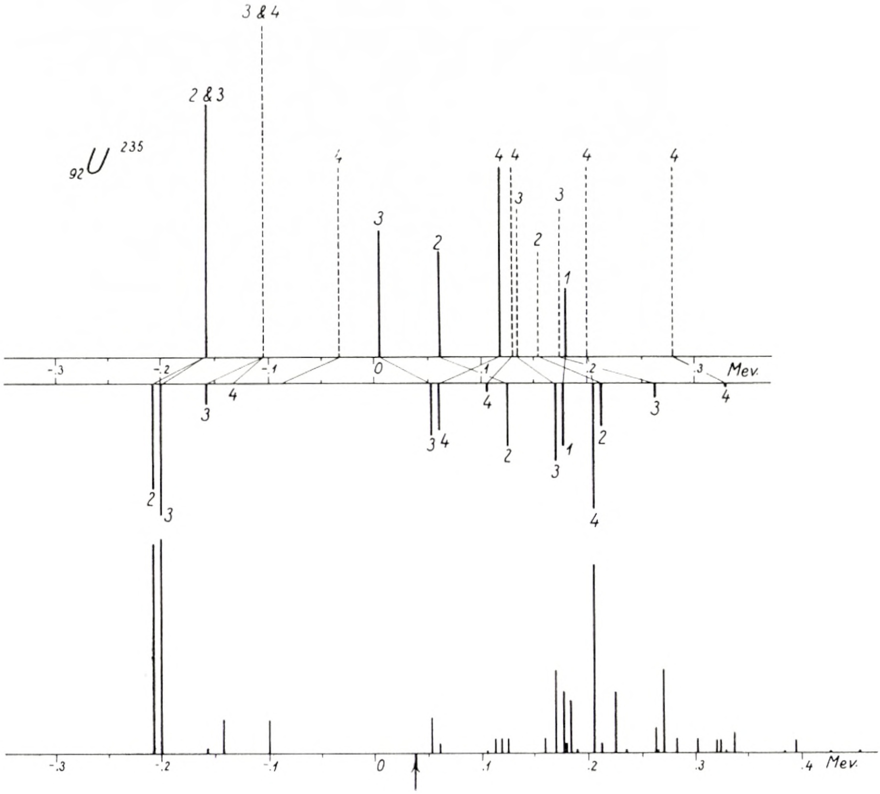

FIGURE 3.

Energy levels and spectrum for the odd isotope ${}_{92}U^{235}$.

The diagrams are essentially the same as those in Fig. 2 (the triplet of quantum numbers (IjF) is replaced by simply F in the first diagram for reasons of simplicity. The dashed lines in the first diagram represent those states with excited nuclei which can mix with the states which have $I = I_0$. The energies in the first diagram include the diagonal (static) quadrupole interactions, but not the non-diagonal interaction, which are included in the second diagram. The scale for the length of the lines in the spectrum is expanded to twice the scale used in the energy level diagrams. The energies were calculated in the strong coupling approximation, using the same parameters as for ${}_{90}Th^{230}$, except for the ground state nuclear spin:

$$E_{fs} = 0.242 \text{ Mev},$$

$$E_{ROT} = \frac{0.050}{6}\left[I(I+1) - I_0(I_0+1)\right]^{\text{Mev.}}, \quad I_0 = 5/2,$$

$$\varepsilon = 0.0986 \text{ Mev } (Q_0 \approx 2.6 \text{ barns}, \; f_q \approx 0.286).$$

Probability that the nucleus be left in the state $I = 7/2$ is 0.45, $I = 9/2$ is 0.06, and $I = 11/2$ is 0.02.

The states with $\Omega_\mu = \pm 3/2$ are given simply by

$$\left(H_{00} + \tfrac{1}{3} E_{fs} - \varepsilon + \frac{\hbar^2}{2\,\Im}\{F(F+1) - 17/2 \mp 15/2\}\right) \cdots |5/2, 3/2 \pm 3/2; FM\rangle_s. \quad (27)$$

In Fig. 3, the level structure and line intensities are given for U^{235}; the parameters are described in the caption. Perhaps the most striking feature of the spectrum is its complexity. It is to be noted, however, that there appear to be two major components of the spectrum and that the separation of these components is a measure of the interaction.

VI. Conclusions.

For nuclei with large deformations, such as are encountered for $155 < A < 185$ and $A > 225$, the interaction of a μ-meson with the rotational states of a nucleus produces splittings of the $2\,p$ atomic levels which are comparable in size with the mesonic fine structure splittings. The effect increases the number of lines observed and influences the general pattern of the spectrum even when individual lines are not resolvable. There is a large probability that the nucleus be left in an excited rotational level after the meson reaches the atomic ground state, with the subsequent emission of a nuclear γ-ray.

Experimental studies of these effects yield directly (for nuclei with large deformations) the quantity $Q_0 \langle j \,\|\, f(r) \,\|\, j'\rangle$ which is a weighted integral of the quadrupole charge density of the nucleus. This gives information about the magnitude and sign of intrinsic nuclear quadrupole moments and nuclear charge distributions. In particular, it provides a method of determining the sign of the intrinsic quadrupole moments of even-even nuclei, a quantity not available previously from other experiments.

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